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The physical projector and topological quantum field theories: $U(1)$ Chern–Simons theory in 2 + 1 dimensions

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Abstract. The recently proposed physical projector approach to the quantization of gauge-invariant systems is applied to the $U(1)$ Chern–Simons theory in 2 + 1 dimensions as one of the simplest examples of a topological quantum field theory. The physical projector is explicitly demonstrated to be capable of effecting the required projection from the initially infinite number of degrees of freedom to the finite set of gauge-invariant physical states whose properties are determined by the topology of the underlying manifold.

1. Introduction

The general gauge invariance principle pervades all of modern physics at the turn of this century, as a most basic conceptual principle unifying the fields of algebra, topology and geometry with those of the fundamental interactions of the elementary quantum excitations in the natural Universe. This fascinating convergence of ideas is probably nowhere better demonstrated than within the recent developments of M-theory as the prime (and sole) candidate for a fundamental unification†. Given the many mathematics and physics riches hidden deep within the structures and dynamics of gauge-invariant theories, a thorough understanding sets a genuine challenge to the methods developed over the years in order to address such issues. For example, a manifest realization of the gauge invariance principle requires the presence among the degrees of freedom of such systems of redundant variables whose dynamics is specified through arbitrary functions characterizing the gauge freedom inherent to the description. This situation leads to specific problems, especially when quantizing such theories, since some gauge-fixing procedure has to be applied in order to effectively remove in a consistent way the contributions of gauge-variant states to physical observables. More often than not, such gauge fixings suffer Gribov problems [2–4], which must be properly addressed if one is to correctly account for the quantum dynamics of gauge-invariant systems, certainly within a non-perturbative framework. Among gauge theories, topological quantum field theories [5, 6] provide the most extreme example of such a situation, since their infinite number of degrees of freedom includes only a *finite* number of gauge-invariant physical states, whose properties are in addition solely determined by the topology of the underlying manifold irrespective of its geometry.

† For a recent discussion, see for example [1].

In a recent development [7], a new approach to the quantization of gauge theories was proposed, which avoids from the outset any gauge fixing and thus any issue of the eventuality of some Gribov problem [8]. This approach is directly set within the necessary framework of Dirac's quantization of constrained systems†. No extension—with its cortège of ghosts and ghosts for ghosts—or reduction of the original set of degrees of freedom is required, as is the case for all approaches which necessarily implement some gauge-fixing procedure with its inherent risks of Gribov problems [4]. Nonetheless, the correct representation of the true quantum dynamics of the system is achieved in this new approach, which uses in an essential way the projection operator [7] onto the subspace of gauge-invariant physical states of a given gauge-invariant system. Some of the advantages of the physical projector approach have already been explored and demonstrated in a few simple quantum mechanical gauge-invariant systems [9, 10]. In this paper, we wish to illustrate how the same methods are capable of also dealing with the intricacies of topological quantum field theories which, even though possessing only a finite set of physical states, require an infinite number of degrees of freedom and states for their formulation. Indeed, it will be shown that the physical projector precisely effects this required projection.

The specific case addressed here as one of the simplest possibilities, is that of the pure Chern–Simons theory in 2 + 1 dimensions with gauge group $U(1)$. Moreover, the discussion will be made explicit when the topology of the underlying manifold is that of $\Sigma \times \mathbb{R}$, where Σ is a two-dimensional compact Riemann surface taken to be a 2-torus T_2 for most of our considerations. This system has been studied from quite a few different points of view [6, 11–15]. The consistency of the physical projector approach will be demonstrated by again deriving some of the same results through an explicit resolution of the gauge-invariant quantum dynamics within this specific framework which avoids any gauge fixing whatsoever and thus also any Gribov problem.

The outline of the discussion is as follows. Section 2 briefly elaborates on the classical constrained Hamiltonian formulation for Chern–Simons theories with an arbitrary gauge symmetry group. These considerations are then particularized in section 3 to the $U(1)$ case restricted to the $\mathbb{R} \times T_2$ topology, enabling a straightforward Fourier mode analysis of the then discrete infinite set of degrees of freedom. In section 4, the Dirac quantization of the system is developed, leading in section 5 to the construction of the physical projector. These results are then explicitly used in section 6 in order to identify the spectrum of physical states in the $U(1)$ theory and to determine their coherent-state wavefunction representations. Finally, the discussion ends with conclusions, while some necessary details are included in an appendix in order not to detract from the main line of arguments.

2. Classical Chern–Simons theories

Let G be a compact simple Lie group of Hermitian generators T^a ($a = 1, 2, \ldots, \dim G$) and structure constants f^{abc} such that $[T^a, T^b] = \mathrm{i} f^{abc} T^c$. In terms of the gauge connection A^a_μ, the action for the associated (2 + 1)-dimensional Chern–Simons theory is then given by

$$\begin{aligned} S &= N_k \int_{\mathbb{R}\times\Sigma} \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\epsilon^{\mu\nu\rho}\big[A^a_\mu \partial_\nu A^a_\rho - \tfrac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho\big] \\ &= \tfrac{1}{2} N_k \int_{\mathbb{R}\times\Sigma} \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\epsilon^{\mu\nu\rho}\big[A^a_\mu F^a_{\nu\rho} + \tfrac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho\big] \end{aligned} \qquad (1)$$

† For a detailed discussion and references to the original literature, see for example [4].

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc} A^b_\mu A^c_\nu$, $\epsilon^{012} = +1$, while N_k is a normalization factor (the usual gauge coupling constant g has been absorbed into the gauge connection A^a_μ).

As is well known, under small gauge transformations (i.e. continuously connected to the identity transformation), the Lagrangian density in (1) remains invariant up to a surface term. For large gauge transformations, however (i.e. those transformations in a homotopy class different from that of the identity transformation), the action (1) changes by a term proportional to a topological invariant, namely the winding number of the gauge transformation. Consequently, at the quantum level, invariance under large gauge transformations requires that the normalization factor N_k be quantized, which is the reason for its notation. Note also that no metric structure whatsoever is necessary for the definition of the action (1). Indices $\mu, \nu, \rho = 0, 1, 2$ are neither raised nor lowered, while the notation, though reminiscent of a Minkowski signature metric, is in fact related to the specific topology $\mathbb{R} \times \Sigma$ considered for the three-dimensional manifold. The choice of the real line $\mathbb{R}$ for the time evolution coordinate x^0 is made for the purpose of canonical quantization hereafter, while any other type of compact topology for the three-dimensional manifold may be obtained from $\mathbb{R} \times \Sigma$ through gluing and twisting [6].

Hence, equation (1) defines a topological field theory, namely a field theory whose gauge freedom is so large that its gauge-invariant configurations characterize solely the topology of the underlying manifold, irrespective of its geometry [5]. In the present case, this is made obvious in terms of the associated equations of motion,

$$\epsilon^{\mu\nu\rho} F^a_{\nu\rho} = 0 \quad \Leftrightarrow \quad F^a_{\mu\nu} = 0. \tag{2}$$

Indeed, the modular space of flat gauge connections on the base manifold is finite dimensional and purely topological in its characterization through holonomies around the non-contractible cycles in the manifold. Quantization of the Chern–Simons theory thus defines the quantization of a system whose configuration space—which actually coincides with its phase space—is this modular space of flat connections.

The action (1) being of first-order form in time derivatives of fields, is already in the Hamiltonian form necessary for canonical quantization [4, 16]. Indeed, we explicitly have,

$$S = \int \mathrm{d}^3x^\mu \left[\partial_0 A^a_i \, N_k \epsilon^{ij} A^a_j + A^a_0 \, N_k \epsilon^{ij} F^a_{ij} - \partial_i \left(N_k \epsilon^{ij} A^a_j A^a_0\right)\right] \tag{3}$$

where ϵ^{ij} $(i, j = 1, 2)$ is the two-dimensional antisymmetric symbol with $\epsilon^{12} = +1$. Consequently, the actual phase space of the system consists of the field components A^a_i $(i = 1, 2)$ which form a pair of conjugate variables with symplectic structure defined by the brackets

$$\{A^a_1(\vec{x}, x^0), A^b_2(\vec{y}, x^0)\} = \frac{1}{2N_k} \, \delta^{ab} \, \delta^{(2)}(\vec{x} - \vec{y}). \tag{4}$$

In addition, the first-class Hamiltonian of the system vanishes identically, $H = 0$, as befits any system invariant under local coordinate reparametrizations, while finally the time components A^a_0 of the gauge connection are the Lagrange multipliers for the first-class constraints

$$\phi^a = -2N_k F^a_{12} = -2N_k \left[\partial_1 A^a_2 - \partial_2 A^a_1 - f^{abc} A^b_1 A^c_2\right] \tag{5}$$

whose algebra of brackets is that of the gauge group G,

$$\{\phi^a(\vec{x}, x^0), \phi^b(\vec{y}, x^0)\} = f^{abc} \phi^c(\vec{x}, x^0) \delta^{(2)}(\vec{x} - \vec{y}). \tag{6}$$

That these constraints are indeed the local generators of small gauge transformations is confirmed through their infinitesimal action on the phase space variables A_i^a,

$$\delta_\theta A_i^a(\vec{x}, x^0) = \left\{ A_i^a(\vec{x}, x^0), \int_\Sigma \mathrm{d}^2\vec{y}\, \theta^b(\vec{y}, x^0)\phi^b(\vec{y}, x^0) \right\}$$
$$= \partial_i\theta^a(\vec{x}, x^0) + f^{abc}\theta^b(\vec{x}, x^0)A_i^c(\vec{x}, x^0) \tag{7}$$

while the Lagrange multipliers A_0^a must vary according to

$$\delta_\theta A_0^a(\vec{x}, x^0) = \partial_0\theta^a(\vec{x}, x^0) + f^{abc}\theta^b(\vec{x}, x^0)A_0^c(\vec{x}, x^0). \tag{8}$$

The constraints ϕ^a also coincide, up to surface terms, with the Noether charge densities related to the gauge symmetry. Indeed, the Noether currents $\gamma^{a\mu} = \epsilon^{\mu\nu\rho}N_k f^{abc}A_\nu^b A_\rho^c$ are conserved for solutions to the equations of motion, $\partial_\mu\gamma^{a\mu} = 0$, so that the associated charges read

$$Q^a = \int_\Sigma \mathrm{d}^2\vec{x}\, \gamma^{a(\mu=0)} = \int_\Sigma \mathrm{d}^2\vec{x} \left[\phi^a + 2N_k(\partial_1 A_2^a - \partial_2 A_1^a)\right]. \tag{9}$$

The above conclusions based on the first-order action (3) are of course confirmed through an explicit application of Dirac's algorithm for the construction of the Hamiltonian formulation of constrained systems [4]. In particular, the brackets (4) then correspond to Dirac brackets after second-class constraints have been solved for, while (3) then describes the so-called 'fundamental Hamiltonian formulation' [4] of any dynamical system. Incidentally, note that surface terms which appear in (3) and (9) are irrelevant for such a Hamiltonian construction, which is in essence a local construct on the manifold Σ. In any case, they do not contribute when Σ is without boundaries.

3. The $U(1)$ theory on the torus

Henceforth, we shall restrict the discussion to the gauge group $G = U(1)$ and to the compact Riemann manifold Σ being the two-dimensional torus T_2. This choice is made for the specific purpose of demonstrating that the physical projector approach is capable of properly quantizing such theories in the simplest of cases, leaving more general choices to be explored elsewhere with the same techniques. In particular, the torus mode expansions to be specified presently may be extended to Riemann surfaces Σ of arbitrary genus through the use of Abelian differentials and the Krichever–Novikov operator formalism [11–13, 17–19]. The extension to non-Abelian gauge groups G requires further techniques of coherent-states not included in the present discussion.

Given the manifold $\Sigma = T_2$, let us consider the local trivialization of this topology associated with a choice of basis of its first homology group with cycles a and b. Correspondingly, the choice of local coordinates x^1 and x^2 is such that $0 < x^1, x^2 < 1$. Related to this trivialization of T_2, fields over T_2 may be Fourier expanded, so that the total number of degrees of freedom, though infinite, is represented in terms of a *discrete* set of modes over T_2. Explicitly, in a real parametrization we have (from here on, any dependency on x^0 is left implicit while the single index $a = 1$ for $G = U(1)$ is not displayed),

$$\begin{aligned}A_i(\vec{x}) = \sum_{n_1=0}^{+\infty}\sum_{n_2=0}^{+\infty} A_i^{++}(n_1,n_2)\cos 2\pi n_1 x^1\ \cos 2\pi n_2 x^2 \\ + \sum_{n_1=0}^{+\infty}\sum_{n_2=1}^{+\infty} A_i^{+-}(n_1,n_2)\cos 2\pi n_1 x^1 \sin 2\pi n_2 x^2 \\ + \sum_{n_1=1}^{+\infty}\sum_{n_2=0}^{+\infty} A_i^{-+}(n_1,n_2)\sin 2\pi n_1 x^1\ \cos 2\pi n_2 x^2 \\ + \sum_{n_1=1}^{+\infty}\sum_{n_2=1}^{+\infty} A_i^{--}(n_1,n_2)\sin 2\pi n_1 x^1 \sin 2\pi n_2 x^2.\end{aligned} \tag{10}$$

Let us emphasize that such expansions do not include the terms which would be associated with the following modes: $A_i^{+-}(n_1, 0)$, $A_i^{-+}(0, n_2)$, $A_i^{--}(n_1, 0)$ and $A_i^{--}(0, n_2)$ $(n_1, n_2 = 1, 2, \ldots)$. Similar mode expansions—to which the remark just made also applies—are obtained for any quantity defined over T_2. For the $U(1)$ generator $\phi(\vec{x})$, one has (for a non-Abelian group G, terms bilinear in the $A_i^{\pm\pm}(n_1, n_2)$ modes also contribute, which is the reason for our restriction to $G = U(1)$)

$$\begin{aligned}\phi^{++}(n_1,n_2) &= -4\pi N_k\big[+n_1 A_2^{-+}(n_1,n_2) - n_2 A_1^{+-}(n_1,n_2)\big]\\ \phi^{+-}(n_1,n_2) &= -4\pi N_k\big[+n_1 A_2^{--}(n_1,n_2) + n_2 A_1^{++}(n_1,n_2)\big]\\ \phi^{-+}(n_1,n_2) &= -4\pi N_k\big[-n_1 A_2^{++}(n_1,n_2) - n_2 A_1^{--}(n_1,n_2)\big]\\ \phi^{--}(n_1,n_2) &= -4\pi N_k\big[-n_1 A_2^{+-}(n_1,n_2) + n_2 A_1^{-+}(n_1,n_2)\big].\end{aligned} \tag{11}$$

Since $\{\phi(\vec{x}), \phi(\vec{y})\} = 0$ in the Abelian $U(1)$ case, all the modes $\phi^{\pm\pm}(n_1, n_2)$ have vanishing brackets with one another, while those for the phase space modes $A_i^{\pm\pm}(n_1, n_2)$ are given by

$$\{A_1^{\pm\pm}(n_1,n_2), A_2^{\pm\pm}(m_1,m_2)\} = \frac{2}{N_k} f^{\pm\pm}(n_1,n_2)\,\delta_{n_1,m_1}\,\delta_{n_2,m_2} \tag{12}$$

where

$$\begin{aligned}f^{++}(n_1,n_2) &= \frac{1}{(1+\delta_{n_1,0})(1+\delta_{n_2,0})} \qquad & f^{+-}(n_1,n_2) &= \frac{1-\delta_{n_2,0}}{1+\delta_{n_1,0}}\\ f^{-+}(n_1,n_2) &= \frac{1-\delta_{n_1,0}}{1+\delta_{n_2,0}} & f^{--}(n_1,n_2) &= (1-\delta_{n_1,0})(1-\delta_{n_2,0}).\end{aligned} \tag{13}$$

In order to understand how small and large gauge transformations—the latter not being generated by the first-class constraint $\phi(\vec{x})$—are represented in terms of these mode expansions, let us consider the general gauge transformation of the field A_μ, namely $A'_\mu = A_\mu + \partial_\mu\theta$, associated with the $U(1)$ local phase transformation $U(\vec{x}, x^0) = \mathrm{e}^{\mathrm{i}\theta(\vec{x},x^0)}$. A point central to our discussion is that the arbitrary function $\theta(\vec{x}, x^0)$ may always be expressed as

$$\theta(\vec{x}, x^0) = \theta_0(\vec{x}, x^0) + 2\pi k_1 x^1 + 2\pi k_2 x^2 \tag{14}$$

where $\theta_0(\vec{x}, x^0)$ is an arbitrary *periodic* function, i.e. a *scalar field* on T_2, while k_1 and k_2 are arbitrary positive or negative integers. Indeed, any small gauge transformation is defined in terms of some function $\theta_0(\vec{x}, x^0)$ with $(k_1, k_2) = (0, 0)$, while any large gauge transformation $\theta(\vec{x}, x^0)$ may always be brought to the above general form with some specific function $\theta_0(\vec{x}, x^0)$, the integers k_1 and k_2 then labelling the $U(1)$ holonomies of the gauge transformation around the chosen a and b homology cycles in T_2. Any gauge transformation thus falls into such a (k_1, k_2) homotopy class of the gauge group over T_2. In terms of the mode expansion of the gauge parameter function (14), the non-zero modes $A_i^{\pm\pm}(n_1, n_2)$ ($n_1 \neq 0$ or $n_2 \neq 0$, or both) are transformed according to

$$\begin{aligned}
\Delta A_1^{++}(n_1,n_2) &= +2\pi n_1 \theta_0^{-+}(n_1,n_2) & \Delta A_2^{++}(n_1,n_2) &= +2\pi n_2 \theta_0^{+-}(n_1,n_2) \\
\Delta A_1^{+-}(n_1,n_2) &= +2\pi n_1 \theta_0^{--}(n_1,n_2) & \Delta A_2^{+-}(n_1,n_2) &= -2\pi n_2 \theta_0^{++}(n_1,n_2) \\
\Delta A_1^{-+}(n_1,n_2) &= -2\pi n_1 \theta_0^{++}(n_1,n_2) & \Delta A_2^{-+}(n_1,n_2) &= +2\pi n_2 \theta_0^{--}(n_1,n_2) \\
\Delta A_1^{--}(n_1,n_2) &= -2\pi n_1 \theta_0^{+-}(n_1,n_2) & \Delta A_2^{--}(n_1,n_2) &= -2\pi n_2 \theta_0^{-+}(n_1,n_2)
\end{aligned} \tag{15}$$

where $\Delta A_i^{\pm\pm}(n_1, n_2) = \left(A_i^{\pm\pm}(n_1, n_2)\right)' - A_i^{\pm\pm}(n_1, n_2)$, while the zero modes $A_i^{++}(0, 0)$ transform as

$$\Delta A_1^{++}(0,0) = 2\pi k_1 \qquad \Delta A_2^{++}(0,0) = 2\pi k_2. \tag{16}$$

These different expressions thus nicely establish that small gauge transformations—generated by the first-class constraint $\phi(\vec{x})$—modify only the non-zero modes and that large gauge transformations only affect the zero modes of the gauge connection $A_i(\vec{x})$. Turning the argument around, one thus concludes that the system factorizes into two types of degrees of freedom, namely the non-zero modes $A_i^{\pm\pm}(n_1, n_2)$ ($n_1 \neq 0$ or $n_2 \neq 0$) directly related to small gauge transformations only, and the zero modes $A_i^{++}(0, 0)$ directly related to large gauge transformations only. Moreover, the above expressions also show that it is always possible to set half the non-zero modes to zero by an appropriate small gauge transformation, namely either the $i = 1$ or the $i = 2$ component for each of the modes $A_i^{\pm\pm}(n_1, n_2)$ (the choice of which of these two components is set to zero is left open for the modes with both $n_1 \neq 0$ and $n_2 \neq 0$, but not for those modes for which either $n_1 = 0$ or $n_2 = 0$). Consequently, invariance under small gauge transformations implies that the physical content of the system actually reduces to that of its zero-mode sector $A_i^{++}(0, 0)$ ($i = 1, 2$) only, while the physical content of its non-zero-mode sector is gauge equivalent to the trivial solution $A_i(\vec{x}) = 0$ to the flat connection condition $F_{12}(\vec{x}) = 0$ associated with the vanishing holonomies $(k_1, k_2) = (0, 0)$. That the physics of these systems lies entirely in their zero-mode sector remains valid at the quantum level as well, as is shown hereafter (in fact, this conclusion also extends to Riemann surfaces Σ of arbitrary genus and for any choice of non-Abelian gauge group G [6, 11–14]).

An identical separation also applies to the modes of the gauge parameter function $\theta(\vec{x}, x^0)$. As shown above, the term $(2\pi k_1 x^1 + 2\pi k_2 x^2)$ corresponds to large gauge transformations only, while the contribution $\theta_0(\vec{x}, x^0)$ induces small transformations only, whose zero mode $\theta_0^{++}(0, 0)$ in fact completely decouples. Indeed, the latter mode corresponds to a global phase transformation, which for the real degrees of freedom $A_\mu(\vec{x}, x^0)$ stands for no transformation at all. In other words, in as far as the small gauge parameter function $\theta_0(\vec{x}, x^0)$ is concerned, one could say that its zero mode $\theta_0^{++}(0, 0)$ has, in fact, been traded for the (k_1, k_2) parameters characterizing the holonomies of a large gauge transformation. The zero mode $\theta_0^{++}(0, 0)$ for small gauge parameter functions $\theta_0(\vec{x})$ thus does not enter our considerations in any way whatsoever, and may always be set to zero. Finally, let us simply point out also that the range of each of the non-zero modes $\theta_0^{\pm\pm}(n_1, n_2)$ is the entire real line, running from $-\infty$ to $+\infty$ (as opposed to the zero mode $\theta_0^{++}(0, 0)$ which would have taken its values in the interval $[0, 2\pi]$, for example, had it contributed to gauge symmetries of the system).

It may appear that by using the mode expansion (10), the gauge field $A_i(\vec{x})$ is assumed to obey periodic boundary conditions, which would amount to implicitly assuming that $A_i(\vec{x})$

defines a vector field over T_2 rather than a possibly non-trivial section of a $U(1)$ bundle over T_2, the latter case being associated with the possibility of twisted boundary conditions for the components $A_i(\vec{x})$ [20]. However, the local trivialization of T_2 in terms of the coordinates $0 < x^1, x^2 < 1$ does not define a complete covering of T_2, which requires at least $2^2 = 4$ different overlapping coordinate charts. Moving from one chart to another implies that the gauge connection $A_i(\vec{x})$ also changes by gauge transformations which include large ones associated with twisted boundary conditions. Therefore, by explicitly considering in our definition of the gauge-invariant system large gauge transformations, the possibility of twisted boundary conditions is implicitly included, and the above Fourier mode decomposition of the phase space degrees of freedom $A_i(\vec{x})$ is fully warranted.

4. Dirac quantization

Given the above mode decompositions of the Hamiltonian formulation of the system defined over the torus T_2, its canonical quantization proceeds straightforwardly through the correspondence principle according to which brackets now correspond to commutation relations being set equal to the value of the bracket multiplied by $\mathrm{i}\hbar$. Thus, given (12), the fundamental quantum operators are the modes $\hat{A}_i^{\pm\pm}(n_1, n_2)$ such that

$$[\hat{A}_1^{\pm\pm}(n_1, n_2), \hat{A}_2^{\pm\pm}(m_1, m_2)] = \frac{2\mathrm{i}\hbar}{N_k} f^{\pm\pm}(n_1, n_2)\delta_{n_1,m_1}\delta_{n_2,m_2} \tag{17}$$

with in particular for the zero modes,

$$\left[\hat{A}_1^{++}(0,0), \hat{A}_2^{++}(0,0)\right] = \mathrm{i}\frac{\hbar}{2N_k}. \tag{18}$$

Henceforth, we shall thus assume explicitly that $N_k > 0$, with the understanding that the case when $N_k < 0$ is then obtained simply by interchanging the roles of the coordinates x^1 and x^2.

It is already possible at this stage to determine the number of quantum physical states [6]. Indeed, the relations (15) and (16) show that the actual gauge-invariant phase space degrees of freedom are the zero modes $A_i^{++}(0,0)$ defined up to integer shifts by 2π, while half the non-zero modes—either the $i = 1$ or 2 component, depending on the given mode—may be set to zero through small gauge transformations. In other words, the actual physical phase space of the system is a two-dimensional torus of volume $(2\pi)^2$, an instance of a phase space which is not a cotangent bundle as is usually the case but rather a compact manifold. The quantization of the system thus amounts to quantizing this two-dimensional torus, with the commutation relation (18) in which $\hbar' = \hbar/(2N_k)$ plays the role of an effective Planck constant. In particular, the total number of physical states is thus given by the volume $(2\pi)^2$ of phase space divided by that of each quantum cell $(2\pi\hbar')$ for the degree of freedom $A_1^{++}(0,0)$, namely

$$\frac{(2\pi)^2}{2\pi(\hbar/2N_k)} = \frac{4\pi}{\hbar}N_k. \tag{19}$$

Consequently, the normalization factor N_k ought to be quantized with a value $N_k = \hbar k/(4\pi)$ to be associated with k physical states ($k = 1, 2, \ldots$). Precisely this quantization condition is established hereafter by considering large gauge transformations of the system; this quantization condition will then be specified further later on when considering modular transformations of the underlying torus T_2, which then require the integer k to also be even.

The above commutation relations for the modes $\hat{A}_i^{\pm\pm}(n_1, n_2)$ define an infinite tensor product of Heisenberg algebras. In order to set up a coherent-state representation through creation and annihilation operators associated with this Heisenberg algebra, it is necessary to introduce a complex structure on the initial base manifold T_2. The necessity of introducing some further structure on Σ beyond the purely topological one is also unavoidable in all other quantization frameworks for Chern–Simons theories [6]. In fact, all other approaches require a metric structure on Σ, while it is then shown that the quantized system nevertheless depends only on the complex structure (or conformal class of the metric) on Σ, with the space of gauge-invariant physical states providing a projective representation of the modular group of Σ due to a quantum conformal anomaly [6]. This is how close a topological quantum field theory may come to being purely topological. In the present approach, the necessity of introducing a complex structure over Σ is thus seen to arise from a coherent-state quantization of the system, while, in contradistinction to other quantization methods, it is also gratifying to realize that no further structure is required within this approach since the quantized system should in any case turn out to be independent of any such additional structure.

On T_2, a complex structure is characterized through a complex parameter $\tau = \tau_1 + \mathrm{i}\tau_2$ whose imaginary part is strictly positive ($\tau_2 > 0$), with the modular group $PSL(2, \mathbb{Z})$ of transformations generated by ($T : \tau \to \tau + 1$) and ($S : \tau \to -1/\tau$) defining the classes of inequivalent complex structures under global diffeomorphisms in T_2. Associated with the complex parametrization,

$$z = x^1 + \tau x^2 \qquad \mathrm{d}z\, \mathrm{d}\bar{z} = |\mathrm{d}x^1 + \tau\, \mathrm{d}x^2|^2 = (\mathrm{d}x^1)^2 + 2\tau_1\, \mathrm{d}x^1\, \mathrm{d}x^2 + |\tau|^2(\mathrm{d}x^2)^2 \tag{20}$$

the gauge connection 1-form reads as

$$A = \mathrm{d}x^1 A_1 + \mathrm{d}x^2 A_2 = \mathrm{d}z\, A_z + \mathrm{d}\bar{z}\, A_{\bar{z}} \tag{21}$$

with

$$A_z = \frac{\mathrm{i}}{2\tau_2}[\bar{\tau} A_1 - A_2] \qquad A_{\bar{z}} = -\frac{\mathrm{i}}{2\tau_2}[\tau A_1 - A_2]. \tag{22}$$

Given the choice of complex structure parametrized by τ, the annihilation operators for the quantized system are defined by

$$\alpha^{\pm\pm}(n_1, n_2) = \sqrt{\frac{1}{f^{\pm\pm}(n_1, n_2)} \frac{N_k}{4\hbar\tau_2}} \left[-\mathrm{i}\tau \hat{A}_1^{\pm\pm}(n_1, n_2) + \mathrm{i}\hat{A}_2^{\pm\pm}(n_1, n_2)\right] \tag{23}$$

with the creation operators $\alpha^{\pm\pm\dagger}(n_1, n_2)$ simply defined as the adjoint operators of $\alpha^{\pm\pm}(n_1, n_2)$. One has

$$[\alpha^{\pm\pm}(n_1, n_2), \alpha^{\pm\pm\dagger}(m_1, m_2)] = \delta_{n_1,m_1}\delta_{n_2,m_2} \tag{24}$$

while the annihilation (respectively, creation) operators clearly correspond, up to normalization, to the Fourier modes of $\hat{A}_{\bar{z}}(z, \bar{z})$ (respectively, $\hat{A}_z(z, \bar{z})$).

An overcomplete basis of the space of quantum states is then provided by the coherent states,

$$|z^{\pm\pm}(n_1, n_2)\rangle = \prod_{\pm\pm} \prod_{n_1,n_2} \mathrm{e}^{-\frac{1}{2}|z^{\pm\pm}(n_1,n_2)|^2}\, \mathrm{e}^{z^{\pm\pm}(n_1,n_2)\alpha^{\pm\pm\dagger}(n_1,n_2)}\, |0\rangle \tag{25}$$

with the following representation of the unit operator:

$$\mathbb{1} = \int \prod_{\pm\pm} \prod_{n_1,n_2} \frac{\mathrm{d}z^{\pm\pm}(n_1, n_2) \wedge \mathrm{d}\bar{z}^{\pm\pm}(n_1, n_2)}{\pi}\, |z^{\pm\pm}(n_1, n_2)\rangle\langle z^{\pm\pm}(n_1, n_2)| \tag{26}$$

where $z^{\pm\pm}(n_1, n_2)$ are arbitrary complex variables and $|0\rangle$ is the usual Fock vacuum normalized such that $\langle 0|0\rangle = 1$. In particular, the gauge-invariant physical states of the system are those superpositions of these coherent states which are annihilated by the first-class operator $\hat{\phi}(\vec{x})$, namely by all the modes $\hat{\phi}^{\pm\pm}(n_1, n_2)$. However, this restriction does not yet account for invariance under large gauge transformations, which are not generated by the constraint operator $\hat{\phi}(\vec{x})$, a further specification to be addressed in the next section.

One could now proceed and solve for the physical-state conditions $\hat{\phi}(\vec{x})|\psi\rangle = 0$ in terms of the above mode decompositions. However, we shall rather pursue the physical projector path, which will enable us to solve these conditions by at the same time also determining the wavefunctions of the corresponding physical states, and including the constraints which arise from the requirement of invariance under large gauge transformations as well. Nonetheless, let us note that the resolution of the physical-state conditions $\hat{\phi}(\vec{x})|\psi\rangle = 0$ has been given in [11] precisely using the functional coherent-state representation of the algebra of the field degrees of freedom, to which we shall thus compare our results. The approach of [11, 12], however, uses the formal manipulation and resolution of the *functional* differential equations expressing the physical-state conditions in the coherent-state wavefunction representation of the commutation relations for the field operators $\hat{A}_z(z, \bar{z})$ and $\hat{A}_{\bar{z}}(z, \bar{z})$. By working rather in terms of Fourier modes as done here in the case of the torus T_2, such formal manipulations are avoided by having only a *discrete* infinity of such operators, thus leaving only the much less critical issue of evaluating *discrete* infinite products of normalization factors for quantum states, for which ζ-function regularization techniques will be applied (since other regularizations would require some physical scale, and hence some geometry structure to be introduced on T_2).

5. The physical projector

In order to construct the physical projector, which in effect projects out from any state its gauge-variant components by averaging the state over all its gauge transformations and thereby only leaving its gauge-invariant components [7], let us first consider the operator which induces all finite small gauge transformations, namely $\hat{U}(\theta_0) = \exp(\mathrm{i}/\hbar \int_{T_2} \mathrm{d}^2\vec{x}\, \theta_0(\vec{x})\hat{\phi}(\vec{x}))$. In terms of the previous mode representations and definitions, one finds

$$
\begin{aligned}
-\frac{1}{4\pi}\sqrt{\frac{\tau_2}{\hbar N_k}} \int_{T_2} \mathrm{d}^2\vec{x}\, \theta_0(\vec{x})\hat{\phi}(\vec{x}) &= \frac{1}{2\sqrt{2}} \sum_{n_2=1}^{+\infty} \left[+n_2 \theta_0^{+-}(0, n_2)\alpha^{++}(0, n_2) \right] \\
&\quad + \frac{1}{2\sqrt{2}} \sum_{n_1=1}^{+\infty} \left[-n_1 \bar{\tau} \theta_0^{-+}(n_1, 0)\alpha^{++}(n_1, 0) \right] \\
&\quad + \frac{1}{2\sqrt{2}} \sum_{n_2=1}^{+\infty} \left[-n_2 \theta_0^{++}(0, n_2)\alpha^{+-}(0, n_2) \right] \\
&\quad + \frac{1}{2\sqrt{2}} \sum_{n_1=1}^{+\infty} \left[+n_1 \bar{\tau} \theta_0^{++}(n_1, 0)\alpha^{-+}(n_1, 0) \right] \\
&\quad + \frac{1}{4} \sum_{n_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} \left[\left(+n_2 \theta_0^{+-}(n_1, n_2) - n_1 \bar{\tau} \theta_0^{-+}(n_1, n_2) \right) \alpha^{++}(n_1, n_2) \right] \\
&\quad + \frac{1}{4} \sum_{n_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} \left[\left(-n_2 \theta_0^{++}(n_1, n_2) - n_1 \bar{\tau} \theta_0^{--}(n_1, n_2) \right) \alpha^{+-}(n_1, n_2) \right]
\end{aligned}
$$

$$+\frac{1}{4}\sum_{n_1=1}^{+\infty}\sum_{n_2=1}^{+\infty}\left[\left(+n_2\theta_0^{--}(n_1,n_2)+n_1\bar{\tau}\theta_0^{++}(n_1,n_2)\right)\alpha^{-+}(n_1,n_2)\right]$$

$$+\frac{1}{4}\sum_{n_1=1}^{+\infty}\sum_{n_2=1}^{+\infty}\left[\left(-n_2\theta_0^{-+}(n_1,n_2)+n_1\bar{\tau}\theta_0^{+-}(n_1,n_2)\right)\alpha^{--}(n_1,n_2)\right]+\left[\text{h.c.}\right]. \qquad (27)$$

Note that, as it should, the zero mode $\theta_0^{++}(0,0)$ of the gauge parameter function $\theta_0(\vec{x})$ does not appear in this expression, and that the zero-mode operators $\alpha^{++(\dagger)}(0,0)$ do not contribute either, showing once again that only small gauge transformations are generated by the first-class constraint $\phi(\vec{x})$. Moreover, it is possible to verify that the commutators of the quantity in (27) with the modes $\hat{A}_i^{\pm\pm}(n_1,n_2)$ do reproduce the expressions in (15), while leaving the zero modes $\hat{A}_i^{++}(0,0)$ invariant, since we have, using the property in (43),

$$\hat{U}(\theta_0)\,\hat{A}_i^{\pm\pm}(n_1,n_2)\,\hat{U}^{-1}(\theta_0)=\hat{A}_i^{\pm\pm}(n_1,n_2)+\left[\frac{\mathrm{i}}{\hbar}\int_{T_2}\mathrm{d}^2\vec{x}\,\theta_0(\vec{x})\hat{\phi}(\vec{x}),\hat{A}_i^{\pm\pm}(n_1,n_2)\right]. \qquad (28)$$

To determine how to construct the operator which generates the large gauge transformations characterized by the holonomies (k_1,k_2), let us consider the zero mode $A_{\bar{z}}^{++}(0,0)$ (a similar analysis is possible based on $A_z^{++}(0,0)$), whose gauge transformation is given by (see (16) and (22))

$$\Delta A_{\bar{z}}^{++}(0,0)=-\frac{\mathrm{i}}{2\tau_2}\,2\pi[\tau k_1-k_2]. \qquad (29)$$

Consequently, the associated annihilation operator $\alpha^{++}(0,0)$ should transform according to

$$\hat{U}(k_1,k_2)\,\alpha^{++}(0,0)\,\hat{U}^{-1}(k_1,k_2)=\alpha^{++}(0,0)-2\mathrm{i}\pi\sqrt{\frac{N_k}{\hbar\tau_2}}\,[\tau k_1-k_2] \qquad (30)$$

where $\hat{U}(k_1,k_2)$ stands for the operator generating the large gauge transformation of holonomies (k_1,k_2). Therefore, we must have

$$\hat{U}(k_1,k_2)=C(k_1,k_2)\exp\left[\frac{2\mathrm{i}\pi}{\hbar}\sqrt{\frac{\hbar N_k}{\tau_2}}\left\{[\bar{\tau}k_1-k_2]\,\alpha^{++}(0,0)+[\tau k_1-k_2]\,\alpha^{++\dagger}(0,0)\right\}\right] \quad (31)$$

where $C(k_1,k_2)$ is a cocycle factor to be determined presently such that the group composition law is obeyed for large gauge transformations,

$$\hat{U}(\ell_1,\ell_2)\,\hat{U}(k_1,k_2)=\hat{U}(\ell_1+k_1,\ell_2+k_2). \qquad (32)$$

Since—using the property in (43)—the latter constraint translates into the cocycle condition

$$\mathrm{e}^{-4\mathrm{i}\pi^2(N_k/\hbar)[\ell_1k_2-\ell_2k_1]}\,C(\ell_1,\ell_2)\,C(k_1,k_2)=C(\ell_1+k_1,\ell_2+k_2) \qquad (33)$$

a careful analysis shows that the unique solution to this cocycle condition is

$$N_k=\frac{\hbar}{4\pi}k \qquad C(k_1,k_2)=\mathrm{e}^{\mathrm{i}\pi kk_1k_2} \qquad (34)$$

where $k=1,2,\ldots,$ is some positive integer value. This specific result for the normalization factor N_k will thus be assumed henceforth.

These results therefore establish that consistency of the quantized system under the action of large gauge transformations in its zero-mode sector $\hat{A}_i^{++}(0,0)$ requires the quantization of the normalization factor N_k in precisely such a manner that a total of k gauge-invariant physical states are expected to exist within the entire space of quantum states generated by the coherent states constructed above. In addition, the operator $\hat{U}(k_1, k_2)$ associated with the large gauge transformation of holonomy (k_1, k_2) is thereby totally specified, while the operator $\hat{U}(\theta_0) = \exp(\mathrm{i}/\hbar \int_{T_2} \mathrm{d}^2\vec{x}\, \theta_0(\vec{x})\hat{\phi}(\vec{x}))$ associated with small gauge transformations of the parameter function $\theta_0(\vec{x})$ is defined in (27).

Consequently, the projector onto gauge-invariant physical states should simply be constructed by summing over all small and large gauge transformations the action of the operators $\hat{U}(\theta_0)$ and $\hat{U}(k_1, k_2)$ just described. Even though this is straightforward for the large gauge transformations, the summation over the small ones still requires some further specifications [7], stemming from the fact that the spectrum of the non-zero modes of the gauge constraint $\hat{\phi}(\vec{x}) = 0$ is continuous. Since, as was seen previously, the small gauge invariance is such that half the non-zero modes $A_i^{\pm\pm}(n_1, n_2)$ ($n_1 \neq 0$ or $n_2 \neq 0$) may be set to zero, the other half being their conjugate phase space variables, let us discuss this specific issue in the following much simpler situation.

Consider a single-degree-of-freedom system with coordinate $\hat{q}$ and conjugate momentum $\hat{p}$, both Hermitian operators, whose commutation relation is the usual Heisenberg algebra $[\hat{q}, \hat{p}] = \mathrm{i}$, and subject to the first-class constraint $\hat{q} = 0$. As the spectrum of this latter operator is continuous, the proper definition of a projector onto the states satisfying this constraint requires us to consider rather the projector onto those states whose $\hat{q}$ eigenvalues lie within some interval $[-\delta, \delta]$, $\delta > 0$ being a parameter whose value may be as small as required [7]. The latter projector is expressed as

$$\mathbb{E}_\delta \equiv \mathbb{E}[-\delta < q < \delta] = \int_{-\delta}^{\delta} \mathrm{d}q\, |q\rangle\langle q| = \int_{-\infty}^{+\infty} \mathrm{d}\xi\, \mathrm{e}^{\mathrm{i}\xi\hat{q}}\, \frac{\sin(\xi\delta)}{\pi\xi} \tag{35}$$

assuming that the position eigenstates are normalized such that $\langle q_1|q_2\rangle = \delta(q_1 - q_2)$. By construction, one has the required properties,

$$\mathbb{E}_\delta^2 = \mathbb{E}_\delta \qquad \mathbb{E}_\delta^\dagger = \mathbb{E}_\delta. \tag{36}$$

However, one would rather wish to consider the operator singling out the $|q = 0\rangle$ component of any state, namely,

$$\mathbb{E}_0 = |q = 0\rangle\langle q = 0|. \tag{37}$$

Even though this operator is indeed Hermitian, it is not strictly in involution since one has

$$\mathbb{E}_0^2 = \delta(0)\, \mathbb{E}_0. \tag{38}$$

In other words, since the position eigenstates of the $\hat{q}$ operators are non-normalizable, the operator $\mathbb{E}_0$ does not define a projection operator in a strict sense, since even though it projects onto the $|q = 0\rangle$ component, it thereby leads to a non-normalizable state satisfying the constraint $\hat{q} = 0$. Nevertheless, the non-normalizable projector $\mathbb{E}_0$ may be constructed from the well defined one $\mathbb{E}_\delta$ through the following limit:

$$\mathbb{E}_0 = \lim_{\delta\to 0} \frac{1}{2\delta}\, \mathbb{E}_\delta = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\xi}{2\pi}\, \mathrm{e}^{\mathrm{i}\xi\hat{q}}. \tag{39}$$

Hence, associated with the choice of normalization of position eigenstates $\langle q_1|q_2\rangle = \delta(q_1 - q_2)$, the non-normalizable projector $\mathbb{E}_0$ onto the states such that $\hat{q}|\psi\rangle = 0$ is simply represented by the integral operator on the right-hand side of this last identity.

Transcribing these considerations to the $U(1)$ Chern–Simons theory, it should be clear that the (non-normalizable) projector onto the (non-normalizable) gauge-invariant physical states of the system is simply given by (recall that the modes $\theta_0^{++}(0,0)$, $\theta_0^{+-}(n_1,0)$, $\theta_0^{-+}(0,n_2)$, $\theta_0^{--}(n_1,0)$ and $\theta_0^{--}(0,n_2)$ are non-existent)

$$\mathbb{E}_0 = \sum_{k_1,k_2=-\infty}^{+\infty} \hat{U}(k_1,k_2) \prod_{\pm\pm}\prod_{n_1,n_2} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\theta_0^{\pm\pm}(n_1,n_2)}{2\pi}\, \hat{U}(\theta_0). \tag{40}$$

Note that this physical projector also defines the physical evolution operator of the system, since the first-class Hamiltonian operator vanishes identically, $\hat{H}=0$, a consequence of invariance under local coordinate transformations in Σ.

6. Gauge-invariant physical states

The set of physical states of the system should now be identifiable simply by applying the physical projector $\mathbb{E}_0$ onto the entire space of states representing the operator algebra of modes $\hat{A}_i^{\pm\pm}(n_1,n_2)$. Working in the basis (25) of coherent states $|z^{\pm\pm}(n_1,n_2)\rangle$, this is tantamount to considering the diagonal matrix elements $\langle z^{\pm\pm}(n_1,n_2)|\mathbb{E}_0|z^{\pm\pm}(n_1,n_2)\rangle$, since these matrix elements simply reduce to a sum of the physical-state contributions as intermediate states, namely,

$$\begin{aligned}\langle z^{\pm\pm}(n_1,n_2)|\mathbb{E}_0|z^{\pm\pm}(n_1,n_2)\rangle &= \sum_r \langle z^{\pm\pm}(n_1,n_2)|r\rangle\langle r|z^{\pm\pm}(n_1,n_2)\rangle \\ &= \sum_r \left|\langle r|z^{\pm\pm}(n_1,n_2)\rangle\right|^2 \end{aligned}\tag{41}$$

where r is a discrete or continuous index labelling all physical states (only a finite number k of which are expected, of course). Therefore, when having obtained an expression for the diagonal matrix elements $\langle z^{\pm\pm}(n_1,n_2)|\mathbb{E}_0|z^{\pm\pm}(n_1,n_2)\rangle$ as a sum of modulus-squared quantities, the physical states of the system together with their coherent-state wavefunction representations are readily identified up to a physically irrelevant phase factor, knowing that except for the factor $\mathrm{e}^{-|z^{\pm\pm}(n_1,n_2)|^2/2}$ for each mode stemming from the normalization of the coherent states, the wavefunctions $\langle r|z^{\pm\pm}(n_1,n_2)\rangle$ are necessarily functions of the variables $z^{\pm\pm}(n_1,n_2)$ only, but not of their complex conjugate values $\bar{z}^{\pm\pm}(n_1,n_2)$. Had the system possessed some global symmetry beyond the $U(1)$ gauge invariance, the associated different quantum numbers of the physical states could have been used to label them, thereby making it easier to identify them as well as their wavefunctions from the above matrix elements suitably extended to include the action of the global symmetry generators [9]. Note also that trying to extract the same information—beginning with the number of physical states—from the partition function $\mathrm{Tr}\,\mathbb{E}_0$ would be problematic. Indeed, that latter quantity is ill-defined since neither the physical states $|r\rangle$ nor the physical projector $\mathbb{E}_0$ are normalizable quantities.

Given that the exponential arguments appearing in the definition of the gauge operators $\hat{U}(\theta_0)$ and $\hat{U}(k_1,k_2)$ are linear in the creation and annihilation operators $\alpha^{\pm\pm(\dagger)}(n_1,n_2)$, the explicit evaluation of the above diagonal coherent-state matrix elements of the physical projector $\mathbb{E}_0$ is rather straightforward even though not immediate for the modes with $n_1\neq 0$ and $n_2\neq 0$. Let us only give the outline of the calculation for the modes with $n_1=0$ or $n_2=0$. The calculation for the modes with $n_1\neq 0$ and $n_2\neq 0$ is similar but involves some matrix algebra since the integrations over $\theta_0^{++}(n_1,n_2)$ and $\theta_0^{--}(n_1,n_2)$, on the one hand, and $\theta_0^{+-}(n_1,n_2)$ and $\theta_0^{-+}(n_1,n_2)$, on the other, are coupled to one another in each case.

Consider again a single-degree-of-freedom system with creation and annihilation operators $a^\dagger$ and a, respectively, such that $[a,a^\dagger]=1$, together with the associated coherent states

$$|z\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2}\,\mathrm{e}^{za^\dagger}\,|0\rangle \qquad \langle z|z\rangle = 1 \tag{42}$$

where $|0\rangle$ is the usual Fock vacuum normalized as $\langle 0|0\rangle = 1$. Using the identity

$$e^{A+B} = e^{-\frac{1}{2}[A,B]}\, e^{A}\, e^{B} \tag{43}$$

valid for any two operators A and B which commute with their commutator $[A, B]$, as well as the obvious properties that

$$a\,|z\rangle = z\,|z\rangle \qquad e^{i\lambda\theta a}\,|z\rangle = e^{i\lambda\theta z}\,|z\rangle \tag{44}$$

where λ and θ are arbitrary real and complex variables, respectively (in keeping with the notation in (27)), one readily concludes that diagonal coherent-state matrix elements of an exponential operator whose argument is linear in the creation and annihilation operators are simply given by

$$\langle z|e^{i\lambda[\theta a+\bar{\theta}a^{\dagger}]}|z\rangle = e^{-\frac{1}{2}\lambda^2|\theta|^2}\, e^{i\lambda\bar{\theta}\bar{z}}\, e^{i\lambda\theta z}. \tag{45}$$

Consequently, in the case of the matrix elements $\langle z^{\pm\pm}(n_1, n_2)|\mathbb{E}_0|z^{\pm\pm}(n_1, n_2)\rangle$, the integrations over the discrete infinite number of non-zero modes $\theta_0^{\pm\pm}(n_1, n_2)$ ($n_1 \neq 0$ or $n_2 \neq 0$, or both) for small gauge transformations simply correspond to Gaussian integrals for each of these contributions, whose results are then multiplied by one another. As mentioned previously, the ensuing discrete infinite products of normalization factors are handled using ζ-function regularization techniques, rendering these products well defined. Moreover, the discrete infinite products of the remaining Gaussian factors may be expressed as a simple exponential whose argument is given by the integral over the torus T_2 of a local quantity built from the modes $z^{\pm\pm}(n_1, n_2)$ defining the coherent state $|z^{\pm\pm}(n_1, n_2)\rangle$ for which the diagonal matrix element is evaluated.

Similar considerations apply to the contribution from the zero-mode sector $z^{++}(0, 0)$, but the summation over the holonomies (k_1, k_2) is such that no clear separation of terms in the form of (41), corresponding to the separate contributions of the distinct physical states, appears to be feasible. In fact, the evaluation of the zero-mode sector contribution to the relevant matrix elements requires an entirely different approach, which is detailed in the appendix and uses different representations of the zero-mode algebra (18). In particular, it is shown in the appendix that the total number of physical states is indeed equal to the value of the integer k which quantizes the normalization factor N_k in (34), so that the index r introduced in (41) takes the following finite set of values, $r = 0, 1, \ldots, (k-1)$.

In order to give the final result of all of these calculations in a convenient form, let us introduce the following quantities associated with the complex parameters $z^{\pm\pm}(n_1, n_2)$ (see (22) and (23)):

$$A_{\bar{z}}^{\pm\pm}(n_1, n_2) = \sqrt{f^{\pm\pm}(n_1, n_2)\,\frac{4\pi}{k\tau_2}}\,z^{\pm\pm}(n_1, n_2) \tag{46}$$

thereby determining a specific function $A_{\bar{z}}(z, \bar{z})$ through its mode expansion. In turn, let us then introduce the further modes defined in terms of those of $A_{\bar{z}}(z, \bar{z})$,

$$\begin{aligned}
\chi^{++}(n_1, n_2) &= i\frac{\tau_2}{\pi}\,\frac{-n_2 A_{\bar{z}}^{+-}(n_1, n_2) - n_1\tau A_{\bar{z}}^{-+}(n_1, n_2)}{n_1^2\tau^2 - n_2^2}\\
\chi^{+-}(n_1, n_2) &= i\frac{\tau_2}{\pi}\,\frac{+n_2 A_{\bar{z}}^{++}(n_1, n_2) - n_1\tau A_{\bar{z}}^{--}(n_1, n_2)}{n_1^2\tau^2 - n_2^2}\\
\chi^{-+}(n_1, n_2) &= i\frac{\tau_2}{\pi}\,\frac{+n_1\tau A_{\bar{z}}^{++}(n_1, n_2) - n_2 A_{\bar{z}}^{--}(n_1, n_2)}{n_1^2\tau^2 - n_2^2}\\
\chi^{--}(n_1, n_2) &= i\frac{\tau_2}{\pi}\,\frac{+n_1\tau A_{\bar{z}}^{+-}(n_1, n_2) + n_2 A_{\bar{z}}^{-+}(n_1, n_2)}{n_1^2\tau^2 - n_2^2}
\end{aligned} \tag{47}$$

with again the understanding that the zero mode $\chi^{++}(0,0) = 0$ is taken to vanish, and that the would-be non-zero modes $\chi^{+-}(n_1, 0)$, $\chi^{-+}(0, n_2)$, $\chi^{--}(n_1, 0)$ and $\chi^{--}(0, n_2)$ $(n_1, n_2 = 1, 2, \ldots)$ do not appear in the mode expansions.

Correspondingly, the modes of the functions $\partial_{\bar z}\chi(z,\bar z)$ and $\partial_z\chi(z,\bar z)$ are such that on the one hand

$$\partial_{\bar z}\chi(z,\bar z) = A_{\bar z}(z,\bar z) - A_{\bar z}^{++}(0,0) \tag{48}$$

namely that all modes of $\partial_{\bar z}\chi(z,\bar z)$ except for the zero mode $(\partial_{\bar z}\chi)^{++}(0,0) = 0$ coincide with the corresponding ones of $A_{\bar z}(z,\bar z)$, and on the other hand, the non-zero modes of $\partial_z\chi(z,\bar z)$ are given by

$$\begin{aligned}
(\partial_z\chi)^{++}(n_1,n_2) &= \frac{-1}{n_1^2\tau^2 - n_2^2}\left\{\left[n_1^2|\tau|^2 - n_2^2\right]A_{\bar z}^{++}(n_1,n_2) + n_1 n_2(\tau-\bar\tau)A_{\bar z}^{--}(n_1,n_2)\right\}\\
(\partial_z\chi)^{+-}(n_1,n_2) &= \frac{-1}{n_1^2\tau^2 - n_2^2}\left\{\left[n_1^2|\tau|^2 - n_2^2\right]A_{\bar z}^{+-}(n_1,n_2) - n_1 n_2(\tau-\bar\tau)A_{\bar z}^{-+}(n_1,n_2)\right\}\\
(\partial_z\chi)^{-+}(n_1,n_2) &= \frac{-1}{n_1^2\tau^2 - n_2^2}\left\{-n_1 n_2(\tau-\bar\tau)A_{\bar z}^{+-}(n_1,n_2) + \left[n_1^2|\tau|^2 - n_2^2\right]A_{\bar z}^{-+}(n_1,n_2)\right\}\\
(\partial_z\chi)^{--}(n_1,n_2) &= \frac{-1}{n_1^2\tau^2 - n_2^2}\left\{+n_1 n_2(\tau-\bar\tau)A_{\bar z}^{++}(n_1,n_2) + \left[n_1^2|\tau|^2 - n_2^2\right]A_{\bar z}^{--}(n_1,n_2)\right\}.
\end{aligned} \tag{49}$$

An important identity that the modes of the function $\chi(z,\bar z)$ satisfy is the functional relation

$$\partial_{\bar z}(\partial_z\chi) = \partial_z A_{\bar z} = \partial_z(\partial_{\bar z}\chi). \tag{50}$$

In terms of these different quantities, finally the coherent-state wavefunction for each of the k physical states $|r\rangle$ of the system is given by $(r = 0, 1, 2, \ldots, k-1)$,

$$\begin{aligned}
\langle r|A_{\bar z}(z,\bar z)\rangle &\equiv \langle r|z^{\pm\pm}(n_1,n_2)\rangle\\
&= \mathrm{e}^{-(k\tau_2/2\pi)\left(A_{\bar z}^{++}(0,0)\right)^2}\,\frac{1}{\eta(\tau)}\,\Theta\begin{bmatrix} r/k \\ 0\end{bmatrix}\left(-\mathrm{i}\frac{k\tau_2}{\pi}A_{\bar z}^{++}(0,0)\middle|k\tau\right)\\
&\quad\times\exp\left[-\frac{\mathrm{i}k}{4\pi}\int_{T_2}\mathrm{d}z\wedge\mathrm{d}\bar z\ |A_{\bar z}(z,\bar z)|^2\right]\\
&\quad\times\exp\left[\frac{\mathrm{i}k}{4\pi}\int_{T_2}\mathrm{d}z\wedge\mathrm{d}\bar z\ \partial_{\bar z}\chi(z,\bar z)\partial_z\chi(z,\bar z)\right]
\end{aligned} \tag{51}$$

where $\eta(\tau)$ is the Dedekind η-function, $\eta = \mathrm{e}^{\mathrm{i}\pi\tau/12}\prod_{n=1}^{+\infty}\left(1-\mathrm{e}^{2\mathrm{i}\pi n\tau}\right)$, Θ is the torus θ-function with characteristics whose definition is given in the appendix, while the physically irrelevant overall phase factor is set to unity. Note that in the last two exponential factors, the integral $\int_{T_2}\mathrm{d}z\wedge\mathrm{d}\bar z\,|A_{\bar z}(z,\bar z)|^2$ does include the contribution $\exp\left(-k\tau_2|A_{\bar z}^{++}(0,0)|^2/(2\pi)\right)$ from the zero-mode component $A_{\bar z}^{++}(0,0)$ of the coherent state $|z^{\pm\pm}(n_1,n_2)\rangle$, in contradistinction to the second integral $\int_{T_2}\mathrm{d}z\wedge\mathrm{d}\bar z\,\partial_{\bar z}\chi(z,\bar z)\partial_z\chi(z,\bar z)$ which only includes contributions from the non-zero modes $A_{\bar z}^{\pm\pm}(n_1,n_2)$ $(n_1 \neq 0$ or $n_2 \neq 0)$. This point is important when checking gauge and modular invariance properties of these physical wavefunctions.

These wavefunctions coincide with those established in [11] by using a functional representation of the commutations relations of the field degrees of freedom $A_i(\vec x)$ in order to solve for the physical-state condition $\hat\phi(\vec x) = 0$ as well as requiring invariance under large gauge transformations. This identity of results thus demonstrates that the physical projector approach is indeed capable, without any gauge-fixing procedure whatsoever and thereby avoiding the possibility of any Gribov problem, to properly identify the actual gauge-invariant content of a system, even when this requires the projection down to only a finite number of components from an initially infinite number of states.

By construction, the states whose coherent-state wavefunctions are given in (51) are invariant under both large and small gauge transformations. However, this property does not necessarily imply that the wavefunctions themselves are invariant under arbitrary gauge transformations of the variables $A_{\bar{z}}(z, \bar{z})$. To make this point explicit, let us consider the gauge transformations of the physical states, whose invariance should thus imply the following identities:

$$\begin{aligned} \langle r|\hat{U}^{-1}(\theta_0)|A_{\bar{z}}(z,\bar{z})\rangle &= \langle r|A_{\bar{z}}(z,\bar{z})\rangle \\ \langle r|\hat{U}^{-1}(k_1,k_2)|A_{\bar{z}}(z,\bar{z})\rangle &= \langle r|A_{\bar{z}}(z,\bar{z})\rangle \end{aligned} \tag{52}$$

valid both for small and large gauge transformations $\hat{U}(\theta_0)$ and $\hat{U}(k_1, k_2)$ whatever the values of their modes $\theta_0^{\pm\pm}(n_1, n_2)$ or their holonomies (k_1, k_2).

In the case of small gauge transformations, a careful analysis of the contributions to the relevant matrix elements establishes the result,

$$\hat{U}^{-1}(\theta_0)|A_{\bar{z}}(z,\bar{z})\rangle = \exp\left[-\frac{\mathrm{i}k}{4\pi}\int_{T_2}\mathrm{d}z\wedge\mathrm{d}\bar{z}\left[\partial_{\bar{z}}\chi\,\partial_z\theta_0 - \partial_{\bar{z}}\theta_0\overline{\partial_{\bar{z}}\chi}\,\right]\right]|A_{\bar{z}}(z,\bar{z}) + \partial_{\bar{z}}\theta_0(z,\bar{z})\rangle \tag{53}$$

as well as

$$\begin{aligned} \langle r|A_{\bar{z}}(z,\bar{z}) + \partial_{\bar{z}}\theta_0(z,\bar{z})\rangle = \exp&\left[-\frac{\mathrm{i}k}{4\pi}\int_{T_2}\mathrm{d}z\wedge\mathrm{d}\bar{z}\left[(A_{\bar{z}} - \partial_{\bar{z}}\chi)\partial_z\theta_0 + \partial_{\bar{z}}\theta_0(\overline{A_{\bar{z}}} - \partial_z\chi)\right]\right] \\ &\times\langle r|A_{\bar{z}}(z,\bar{z})\rangle \end{aligned} \tag{54}$$

where $A_{\bar{z}}(z, \bar{z}) + \partial_{\bar{z}}\chi(z, \bar{z})$ stands for the transformations of the modes $A_{\bar{z}}^{\pm\pm}(n_1, n_2)$ under the small gauge transformation $\theta_0(\vec{x})$ of modes $\theta_0^{\pm\pm}(n_1, n_2)$, with of course the understanding that the zero mode $A_{\bar{z}}^{++}(0, 0)$ is left invariant.

Combining these two relations and using the identity in (50) after integration by parts in the exponential factors (to which the zero mode $A_{\bar{z}}^{++}(0, 0)$ does not contribute), the first identity in (52) then indeed follows, thereby confirming gauge invariance of the physical states under small gauge transformations.

Similarly for a large gauge transformation $\hat{U}(k_1, k_2)$ of holonomies (k_1, k_2), one finds,

$$\begin{aligned} \hat{U}^{-1}(k_1,k_2)|A_{\bar{z}}(z,\bar{z})\rangle = \mathrm{e}^{\mathrm{i}\pi kk_1k_2}\,&\mathrm{e}^{-\frac{1}{2}\mathrm{i}k[(k_1\tau-k_2)\overline{A_{\bar{z}}^{++}(0,0)}+(k_1\bar{\tau}-k_2)A_{\bar{z}}^{++}(0,0)]} \\ &\times\left|A_{\bar{z}}(z,\bar{z}) - \frac{\mathrm{i}\pi}{\tau_2}(\tau k_1 - k_2)\right\rangle \end{aligned} \tag{55}$$

as well as

$$\langle r|A_{\bar{z}}(z,\bar{z}) - \frac{\mathrm{i}\pi}{\tau_2}(\tau k_1 - k_2)\rangle = \mathrm{e}^{-\mathrm{i}\pi kk_1k_2}\,\mathrm{e}^{\frac{1}{2}\mathrm{i}k[(k_1\tau-k_2)\overline{A_{\bar{z}}^{++}(0,0)}+(k_1\bar{\tau}-k_2)A_{\bar{z}}^{++}(0,0)]}\,\langle r|A_{\bar{z}}(z,\bar{z})\rangle \tag{56}$$

where this time $A_{\bar{z}}(z, \bar{z}) - \mathrm{i}\pi(\tau k_1 - k_2)/\tau_2$ stands for the transformation of the modes $A_{\bar{z}}^{\pm\pm}(n_1, n_2)$ under the large gauge transformation of holonomies (k_1, k_2) which of course affects only the zero mode $A_{\bar{z}}^{++}(0, 0)$ by the indicated constant shift linear in k_1 and k_2. Consequently, these two relations also lead to the second identity in (52), thereby establishing gauge invariance of the physical states under large gauge transformations as well.

These relations also demonstrate that in spite of the gauge invariance of the physical states $|r\rangle$ $(r = 0, 1, \ldots, k-1)$ under small and large transformations, $\hat{U}(\theta_0)|r\rangle = |r\rangle$

and $\hat{U}(k_1, k_2)|r\rangle = |r\rangle$, their coherent-state wavefunctions $\langle r|A_{\bar{z}}(z, \bar{z})\rangle$ are not invariant by themselves under the simple substitution of the gauge variation of their argument $A_{\bar{z}}(z, \bar{z})$, namely the variations $A'_{\bar{z}}(z, \bar{z}) = A_{\bar{z}}(z, \bar{z}) + \partial_{\bar{z}}\chi(z, \bar{z})$ for small gauge transformations and $A'_{\bar{z}}(z, \bar{z}) = A_{\bar{z}}(z, \bar{z}) - \mathrm{i}\pi(\tau k_1 - k_2)/\tau_2$ for large ones.

Let us now turn to the issue of modular invariance of the physical content of the quantized system. If indeed this content depends only on the complex structure—parametrized by the variable τ—as the only structure necessary beyond the mere topological one of the underlying torus T_2, the spectrum of physical states should remain invariant under the modular group $PSL(2, \mathbb{Z})$ of T_2 which is generated by the transformations

$$T : \tau \to \tau + 1 \qquad S : \tau \to -\frac{1}{\tau}. \tag{57}$$

Indeed, these transformations correspond to global diffeomorphisms, namely Dehn twists, in the local trivialization of T_2 characterized through the choice of holonomy basis (a, b) and the associated coordinates $0 < x^1, x^2 < 1$, and these modular transformations define equivalences classes for the values of τ which correspond to a same complex structure (or conformal class) on T_2.

An explicit analysis of the transformation properties of the physical wavefunctions (51) shows that the requirement of invariance of the physical content of the system under the T modular transformation is met only if the integer k *also takes an even value* (see the appendix) [11], in which case one finds

$$T : \langle r|A_{\bar{z}}(z, \bar{z})\rangle \to \mathrm{e}^{-\mathrm{i}\pi/12}\,\mathrm{e}^{\mathrm{i}\pi r^2/k}\,\langle r|A_{\bar{z}}(z, \bar{z})\rangle. \tag{58}$$

On the other hand, invariance under the S modular transformation is realized through the transformations

$$S : \tau \to \tilde{\tau} = -\frac{1}{\tau} \qquad A^{++}_{\bar{z}}(0, 0) \to \tilde{A}^{++}_{\bar{z}}(0, 0) = -\bar{\tau}\,A^{++}_{\bar{z}}(0, 0) \tag{59}$$

while the non-zero modes $A^{\pm\pm}_{\bar{z}}(n_1, n_2)$ are left unchanged, in which case one has (see the appendix),

$$S : \langle r|A_{\bar{z}}(z, \bar{z})\rangle_\tau \to \langle r|\tilde{A}_{\bar{z}}(z, \bar{z})\rangle_{\tilde{\tau}} = \sum_{r'=0}^{k-1} \frac{1}{\sqrt{k}}\,\mathrm{e}^{2\mathrm{i}\pi r r'/k}\,\langle r'|A_{\bar{z}}(z, \bar{z})\rangle_\tau. \tag{60}$$

Hence, in addition to the quantization condition $N_k = \hbar k/(4\pi)$ imposed on the normalization factor N_k by the requirement of invariance under large gauge transformations, modular invariance of the theory also requires that the integer k be even [11]. In this case the space of quantum physical states does provide an irreducible representation of the T_2 modular group, showing that the physical content of the system does indeed depend only on the choice of complex structure (or conformal class) on T_2 characterized through the equivalence class under the modular group of the parameter τ. Nevertheless, physical states are not individually modular invariant, but rather they define a projective representation of the modular group. This is how close the quantized $U(1)$ Chern–Simons theory is to being a purely topological quantum field theory, the dependency on a complex structure on T_2 following from the existence of a conformal anomaly at the quantum level [6].

7. Conclusions

This paper has demonstrated that the new approach to the quantization of gauge-invariant systems [7], based on the physical projector onto the subspace of gauge-invariant states, is perfectly adequate to handle the intricacies of topological quantum fields theories, in which only a finite set of physical states is to remain after the infinity of gauge-variant configurations has been projected away. This new approach to the quantization of constrained systems does not require any gauge-fixing procedure whatsoever and is thus free of any potential Gribov ambiguity in the case of gauge symmetries [8], in contradistinction to all other quantization frameworks for gauge-invariant systems. Another of its advantages is that the physical projector approach to the quantization of such theories is directly set simply within Dirac's formulation, where it finds its natural place. In particular, the physical projector enables the construction of the physical evolution operator of such systems, to which only physical states contribute as intermediate states, so that the physical content may be identified directly from the matrix elements of that evolution operator, including the wavefunctions of physical states. For topological field theories, since these systems are invariant under local diffeomorphisms in the base manifold, their gauge-invariant Hamiltonian vanishes identically, in which case the evolution operator coincides with the physical projector.

More specifically, the physical projector approach was applied to the $U(1)$ pure Chern–Simons theory in 2 + 1 dimensions in a space whose topology is that of $\mathbb{R} \times T_2$, where T_2 is an arbitrary two-dimensional torus. Through a careful analysis of the quantized system, of the relevant physical projector, and in particular of the specific discrete infinite mode content of the system, it has been possible to identify and construct the physical spectrum and the coherent-state wavefunctions of the gauge-invariant states, leading to results which are in complete agreement with those of other approaches to the quantization of the same system [6, 11–14], while also avoiding some of complications or formal manipulations inherent to these other approaches. The only more or less *ad hoc* but unavoidable feature which is introduced in our analysis is that the discrete infinite products—rather than continuous ones as occurs in functional representations—of Gaussian normalization factors have been evaluated using ζ-function regularization, which avoids having to introduce any other structure on the underlying two-dimensional Riemann surface beyond those associated already to the topology and complex structure (or conformal class) of that surface. This is in keeping with the fact that the quantized theory only depends on that complex structure but no other structure beyond it (even when one is introduced, namely through a metric structure [6]), while also a dependency on the complex structure rather than purely the topology of the underlying manifold is the unavoidable consequence of the quantization of the system [6]. In particular, it was shown how gauge invariance under large gauge transformations implies a quantization rule for the overall normalization of the classical Chern–Simons action in terms of an integer equal to the number of physical states, which in turn is also required to be even for modular invariance to be realized [11]. When both these restrictions are met, the quantized system indeed only depends on the complex structure introduced on the underlying Riemann surface.

These systems are also distinguished by the fact that their physical phase space is a compact manifold, in the present instance with the topology of a two-dimensional torus, which is in contradistinction to the ordinary situation in which the phase space of a given system is a cotangent bundle. Usually, geometric quantization techniques are then invoked in order to address the specific issues raised by a phase space having a compact topology [21]. Nevertheless, no such techniques were introduced here, but rather by properly identifying the operator which generates the transformations responsible for such a compact topology of phase space—in the present instance large gauge transformations—it was possible to properly represent the consequences of such a circumstance using straightforward coherent-state techniques of ordinary quantum mechanics. Clearly, similar considerations based on the construction of the relevant projection operator are of application to any system whose phase space includes a compact manifold which is a homogeneous coset space G/H, where G and

H are compact Lie algebras. From that point of view, it may well be worthwhile to explore the potential of the physical projector as an alternative to geometric quantization techniques.

The physical projector approach is thus quite an efficient approach to the quantization of gauge-invariant systems, which does not require any gauge-fixing procedure whatsoever and thus avoids the potential Gribov problems inherent to such procedures. Some of its advantages have been illustrated here in the instance of the $U(1)$ Chern–Simons theory, as well as for some simple gauge-invariant quantum mechanical systems elsewhere [9, 10]. Hence, it appears timely now to start exploring the application [22] of this alternative method to the quantization of gauge-invariant systems of more direct physical interest, within the context of the recent developments surrounding M-theory compactified to low dimensions, and aiming beyond that towards the gauge-invariant theories of the fundamental interactions among the elementary quantum excitations in the natural Universe.

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Appendix

This appendix outlines the calculation of the contribution of the zero-mode sector to the coherent-state diagonal matrix elements $\langle z^{\pm\pm}(n_1, n_2)|\mathbb{E}_0|z^{\pm\pm}(n_1, n_2)\rangle$ of the physical projector operator, namely the quantity

$$\sum_{k_1,k_2=-\infty}^{+\infty} \langle z|\hat{U}(k_1, k_2)|z\rangle = \sum_{k_1,k_2=-\infty}^{+\infty} \langle z|\mathrm{e}^{\mathrm{i}\pi k k_1 k_2}\, \mathrm{e}^{\mathrm{i}\sqrt{\pi k/\tau_2}\{[\bar{\tau}k_1-k_2]\alpha+[\tau k_1-k_2]\alpha^\dagger\}}|z\rangle. \tag{A1}$$

Here, z, α and $\alpha^\dagger$ stand of course for the zero-mode sector only, with the obvious understanding that usual indices distinguishing these zero modes, such as $z^{++}(0, 0)$, are not explicitly displayed in this appendix.

Since the evaluation of these matrix elements requires changes of bases for different representations of the associated quantum algebra, let us recall here the relations between the different quantum operators appearing in this sector of the system. One has the definitions

$$\alpha = \frac{1}{2}\sqrt{\frac{k}{\pi\tau_2}}\left[-\mathrm{i}\tau\hat{A}_1 + \mathrm{i}\hat{A}_2\right] \qquad \alpha^\dagger = \frac{1}{2}\sqrt{\frac{k}{\pi\tau_2}}\left[\mathrm{i}\bar{\tau}\hat{A}_1 - \mathrm{i}\hat{A}_2\right] \tag{A2}$$

while the corresponding commutation rules are

$$\left[\hat{A}_1, \hat{A}_2\right] = \frac{2\mathrm{i}\pi}{k} \qquad [\alpha, \alpha^\dagger] = 1 \tag{A3}$$

with the implicit understanding that both $\hat{A}_1$ and $\hat{A}_2$ are Hermitian operators. Using these relations as well as the identity in (43), large gauge transformations also read as

$$\hat{U}(k_1, k_2) = \mathrm{e}^{\mathrm{i}\pi k k_1 k_2}\, \mathrm{e}^{\mathrm{i}k[k_1\hat{A}_2 - k_2\hat{A}_1]} = \mathrm{e}^{\mathrm{i}k k_1\hat{A}_2}\, \mathrm{e}^{-\mathrm{i}k k_2\hat{A}_1}. \tag{A4}$$

Since the algebra for the modes $\hat{A}_1$ and $\hat{A}_2$ is that of the usual Heisenberg algebra, with $\hat{A}_1$ playing the role of the configuration-space coordinate and $\hat{A}_2$ that of the conjugate momentum variable, it is clear from the last expression above of the operator $\hat{U}(k_1, k_2)$ in terms of these modes, that the mixed configuration–momentum-space matrix elements of $\hat{U}(k_1, k_2)$ are readily obtained. Hence, let us develop the different representations of the commutations relations (A3), namely the configuration space, the momentum space and the coherent state ones.

Configuration and momentum-space representations correspond to eigenstates $|A_1\rangle$ and $|A_2\rangle$ of the $\hat{A}_1$ and $\hat{A}_2$ operators, respectively,

$$\hat{A}_1|A_1\rangle = A_1|A_1\rangle \qquad \hat{A}_2|A_2\rangle = A_2|A_2\rangle \tag{A5}$$

whose normalization is chosen to be such that

$$\langle A_1|A_1'\rangle = \delta(A_1 - A_1') \qquad \langle A_2|A_2'\rangle = \delta(A_2 - A_2') \tag{A6}$$

to which the following representations of the identity operator are thus associated:

$$\mathbb{1} = \int_{-\infty}^{+\infty} \mathrm{d}A_1\,|A_1\rangle\langle A_1| \qquad \mathbb{1} = \int_{-\infty}^{+\infty} \mathrm{d}A_2\,|A_2\rangle\langle A_2|. \tag{A7}$$

Since the factor $\hbar' = 2\pi/k$ plays the role of an effective Planck constant, it is clear that the configuration-space wavefunction representations of the two operators $\hat{A}_1$ and $\hat{A}_2$ are simply,

$$\langle A_1|\hat{A}_1|\psi\rangle = A_1\,\langle A_1|\psi\rangle \qquad \langle A_1|\hat{A}_2|\psi\rangle = -\frac{2\mathrm{i}\pi}{k}\,\frac{\partial}{\partial A_1}\,\langle A_1|\psi\rangle \tag{A8}$$

and for the momentum space wavefunction representations,

$$\langle A_2|\hat{A}_1|\psi\rangle = \frac{2\mathrm{i}\pi}{k}\,\frac{\partial}{\partial A_2}\,\langle A_2|\psi\rangle \qquad \langle A_2|\hat{A}_2|\psi\rangle = A_2\,\langle A_2|\psi\rangle. \tag{A9}$$

In particular, given the above choice of normalization, we have for the matrix elements expressing the corresponding changes of basis,

$$\langle A_2|A_1\rangle = \frac{\sqrt{k}}{2\pi}\,\mathrm{e}^{-(\mathrm{i}k/2\pi)A_1A_2} \qquad \langle A_1|A_2\rangle = \frac{\sqrt{k}}{2\pi}\,\mathrm{e}^{+(\mathrm{i}k/2\pi)A_1A_2}. \tag{A10}$$

Let us now consider the Fock state representation of the same quantum algebra, whose set of orthonormalized basis vectors is thus defined by

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\alpha^\dagger\right)^n|0\rangle \tag{A11}$$

where $|0\rangle$ is of course the Fock vacuum normalized such that $\langle 0|0\rangle = 1$. Given the above configuration-space representation, the configuration-space wavefunctions of the Fock basis vectors are easily constructed. For the vacuum, one finds from the condition $\alpha|0\rangle = 0$, including proper normalization,

$$\langle A_1|0\rangle = \left(\frac{k\tau_2}{2\pi^2}\right)^{1/4}\,\mathrm{e}^{(\mathrm{i}k/4\pi)\tau A_1^2} \tag{A12}$$

while the excited Fock states are such that

$$\langle A_1|n\rangle = \left(\frac{k\tau_2}{2\pi^2}\right)^{1/4}\,\frac{1}{\sqrt{n!}}\,\left(\frac{\mathrm{i}\bar{\tau}}{2\tau_2}\right)^{n/2}\left[u - \frac{\mathrm{d}}{\mathrm{d}u}\right]^n\,\mathrm{e}^{\frac{1}{2}(\tau/\bar{\tau})u^2} \qquad u = \sqrt{\frac{\mathrm{i}k\bar{\tau}}{2\pi}}\,A_1. \tag{A13}$$

In order to solve for these latter expressions, let us introduce polynomials $P_n(u;\lambda)$ generalizing the usual Hermite polynomials, and defined by the generating function

$$\mathrm{e}^{-\frac{1}{2}(1+\lambda)t^2+(1+\lambda)tu} = \sum_{n=0}^{+\infty}\,\frac{t^n}{n!}\,P_n(u;\lambda) \tag{A14}$$

where λ is a complex parameter. These polynomials satisfy the following set of properties:

$$P_n(u;\lambda) = \mathrm{e}^{\frac{1}{2}\lambda u^2}\left[u - \frac{\mathrm{d}}{\mathrm{d}u}\right]^n \mathrm{e}^{-\frac{1}{2}\lambda u^2} \tag{A15}$$

$$P_{n+1}(u;\lambda) = (1+\lambda)uP_n(u;\lambda) - \frac{\mathrm{d}}{\mathrm{d}u}P_n(u;\lambda) \tag{A16}$$

$$P_0(u;\lambda) = 1 \qquad P_1(u;\lambda) = (1+\lambda)u. \tag{A17}$$

Hence, finally, one finds

$$\langle A_1|n\rangle = \left(\frac{k\tau_2}{2\pi^2}\right)^{1/4} \frac{1}{\sqrt{n!}} \left(\frac{\mathrm{i}\bar\tau}{2\tau_2}\right)^{n/2} \mathrm{e}^{(\mathrm{i}k\tau/4\pi)A_1^2}\, P_n\left(\sqrt{\frac{\mathrm{i}k\bar\tau}{2\pi}}\,A_1; \frac{-\mathrm{i}\tau}{\mathrm{i}\bar\tau}\right). \tag{A18}$$

Similarly for the momentum wavefunctions, one has

$$\langle A_2|n\rangle = \left(\frac{k\tau_2}{2\pi^2|\tau|^2}\right)^{1/4} \frac{(-\mathrm{i})^n}{\sqrt{n!}} \left(\frac{\mathrm{i}\bar\tau}{2\tau_2}\right)^{n/2} \mathrm{e}^{(k/4\mathrm{i}\pi\tau)A_2^2}\, P_n\left(\sqrt{\frac{k}{2\mathrm{i}\pi\bar\tau}}\,A_2; \frac{\mathrm{i}\bar\tau}{-\mathrm{i}\tau}\right). \tag{A19}$$

Finally, let us consider the coherent-state basis,

$$|z\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2}\,\mathrm{e}^{z\alpha^\dagger}\,|0\rangle \qquad \mathbb{1} = \int \frac{\mathrm{d}z\wedge\mathrm{d}\bar z}{\pi}\,|z\rangle\langle z|. \tag{A20}$$

Using the relations

$$\alpha|z\rangle = z|z\rangle \qquad \langle n|z\rangle = \frac{z^n}{\sqrt{n!}}\,\mathrm{e}^{-\frac{1}{2}|z|^2} \tag{A21}$$

as well as the above generating function for the polynomials $P_n(u;\lambda)$ which appear in the matrix elements $\langle A_1|n\rangle$ and $\langle A_2|n\rangle$, the following results are readily obtained:

$$\langle A_1|z\rangle = \left(\frac{k\tau_2}{2\pi^2}\right)^{1/4} \mathrm{e}^{-\frac{1}{2}|z|^2}\,\mathrm{e}^{(\mathrm{i}k\tau/4\pi)A_1^2}\,\mathrm{e}^{-\frac{1}{2}z^2+zA_1\sqrt{(k\tau_2/\pi)}} \tag{A22}$$

$$\langle A_2|z\rangle = \left(\frac{k\tau_2}{2\pi^2|\tau|^2}\right)^{1/4} \mathrm{e}^{-\frac{1}{2}|z|^2}\,\mathrm{e}^{(k/4\mathrm{i}\pi\tau)A_2^2}\,\mathrm{e}^{\frac{1}{2}(\mathrm{i}\bar\tau/-\mathrm{i}\tau)z^2+(zA_2/\tau)\sqrt{k\tau_2/\pi}}. \tag{A23}$$

Having established these different changes of bases, let us return to the evaluation of the matrix element (A1). Obviously, given (A4), the mixed configuration–momentum space matrix elements are simply

$$\langle A_2|\sum_{k_1,k_2=-\infty}^{+\infty}\hat U(k_1,k_2)|A_1\rangle = \sum_{k_1,k_2=-\infty}^{+\infty}\frac{\sqrt{k}}{2\pi}\,\mathrm{e}^{\mathrm{i}kk_1A_2}\,\mathrm{e}^{-\mathrm{i}kk_2A_1}\,\mathrm{e}^{-(\mathrm{i}k/2\pi)A_1A_2}. \tag{A24}$$

Using the identities

$$\sum_{k_1=-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}kk_1A_2} = \sum_{n_1=-\infty}^{+\infty}\frac{2\pi}{k}\delta\left(A_2 - \frac{2\pi n_1}{k}\right) \tag{A25}$$

$$\sum_{k_2=-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}kk_2A_1} = \sum_{n_2=-\infty}^{+\infty}\frac{2\pi}{k}\delta\left(A_1 - \frac{2\pi n_2}{k}\right) \tag{A26}$$

one also has

$$\langle A_2|\sum_{k_1,k_2=-\infty}^{+\infty}\hat{U}(k_1,k_2)|A_1\rangle = \frac{2\pi}{k}\frac{1}{\sqrt{k}}\sum_{n_1,n_2=-\infty}^{+\infty}\mathrm{e}^{-(2\mathrm{i}\pi/k)n_1n_2}\,\delta\left(A_2-\frac{2\pi n_1}{k}\right)\times\delta\left(A_1-\frac{2\pi n_2}{k}\right). \tag{A27}$$

However, in this form it is not yet possible to identify the different contributions of the physical states in the form of modulus-squared terms. In order to achieve that aim, let us finally compute the configuration-space matrix elements of the relevant operator, using the change of basis $\langle A_1|A_2\rangle$ above. One then obtains,

$$\langle A_1|\sum_{k_1,k_2=-\infty}^{+\infty}\hat{U}(k_1,k_2)|A_1'\rangle = \int_{-\infty}^{+\infty}\mathrm{d}A_2\,\langle A_1|A_2\rangle\langle A_2|\sum_{k_1,k_2=-\infty}^{+\infty}\hat{U}(k_1,k_2)|A_1'\rangle$$
$$= \frac{2\pi}{k}\sum_{n_2=-\infty}^{+\infty}\left[\delta\left(A_1'-\frac{2\pi n_2}{k}\right)\sum_{n_1=-\infty}^{+\infty}\delta\left(A_1-\frac{2\pi n_2}{k}-2\pi n_1\right)\right]. \tag{A28}$$

Having thus obtained the configuration-space matrix elements of the zero-mode physical projector operator, let us finally apply it onto any state of this sector of the quantized system. Introducing the configuration-space wavefunction

$$|\psi\rangle = \int_{-\infty}^{+\infty}\mathrm{d}A_1\,|A_1\rangle\langle A_1|\psi\rangle = \int_{-\infty}^{+\infty}\mathrm{d}A_1\,|A_1\rangle\,\psi(A_1) \qquad \psi(A_1)\equiv\langle A_1|\psi\rangle \tag{A29}$$

one readily derives,

$$\langle A_1|\sum_{k_1,k_2=-\infty}^{+\infty}\hat{U}(k_1,k_2)|\psi\rangle = \int_{-\infty}^{+\infty}\mathrm{d}A_1'\,\langle A_1|\sum_{k_1,k_2=-\infty}^{+\infty}\hat{U}(k_1,k_2)|A_1'\rangle\langle A_1'|\psi\rangle$$
$$= \frac{2\pi}{k}\sum_{n_2=-\infty}^{+\infty}\left[\psi\left(\frac{2\pi n_2}{k}\right)\sum_{n_1=-\infty}^{+\infty}\delta\left(A_1-\frac{2\pi n_2}{k}-2\pi n_1\right)\right]. \tag{A30}$$

In particular, since physical states are to be invariant under the action of the physical projector $\sum_{k_1,k_2=-\infty}^{+\infty}\hat{U}(k_1,k_2)$, they should thus obey the following equation:

$$\frac{2\pi}{k}\sum_{n_2=-\infty}^{+\infty}\left[\psi\left(\frac{2\pi n_2}{k}\right)\sum_{n_1=-\infty}^{+\infty}\delta\left(A_1-\frac{2\pi n_2}{k}-2\pi n_1\right)\right] = \psi(A_1). \tag{A31}$$

However, since this equation possesses k distinct linearly independent solutions given by

$$\langle A_1|r\rangle = \psi_r(A_1) = \frac{2\pi}{k}C_r\sum_{n=-\infty}^{+\infty}\delta\left(A_1-\frac{2\pi r}{k}-2\pi n\right) \qquad r=0,1,2,\ldots,k-1 \tag{A32}$$

where C_r is some normalization factor, it is clear that there are exactly k distinct physical states $|r\rangle$ for the quantized $U(1)$ Chern–Simons theory whose action is normalized with the factor $N_k = \hbar k/(4\pi)$. Moreover, the normalization C_r for each of these configuration-space wavefunctions of physical states is obtained from the obvious condition

$$\langle A_1|\sum_{k_1,k_2=-\infty}^{+\infty}\hat{U}(k_1,k_2)|A_1'\rangle = \sum_{r=0}^{k-1}\langle A_1|r\rangle\langle r|A_1'\rangle. \tag{A33}$$

The explicit resolution of this last constraint then finally provides the following configuration-space wavefunctions for the k physical states $|r\rangle$ ($r = 0, 1, 2, \ldots, k-1$):

$$\langle A_1|r\rangle = \sqrt{\frac{2\pi}{k}} \sum_{n=-\infty}^{+\infty} \delta\left(A_1 - \frac{2\pi r}{k} - 2\pi n\right). \tag{A34}$$

The coherent-state wavefunctions $\langle r|z\rangle$ of the same states are then easily obtained, using the associated change of basis specified by the quantities $\langle A_1|z\rangle$, namely

$$\langle r|z\rangle = \int_{-\infty}^{+\infty} \mathrm{d}A_1\, \langle r|A_1\rangle\langle A_1|z\rangle. \tag{A35}$$

An explicit calculation then finds

$$\langle r|z\rangle = \left(\frac{2\tau_2}{k}\right)^{1/4} \mathrm{e}^{-\frac{1}{2}|z|^2}\, \mathrm{e}^{\frac{1}{2}(-\mathrm{i}z)^2}\, \Theta\begin{bmatrix} r/k \\ 0 \end{bmatrix}\left(-\mathrm{i}k\sqrt{\frac{\tau_2}{\pi k}}\, z\middle|k\tau\right) \tag{A36}$$

where the Θ-function with characteristics is defined by [23]

$$\Theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\left(z\middle|\tau\right) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\pi\tau(n+\alpha)^2 + 2\mathrm{i}\pi(n+\alpha)(z+\beta)}. \tag{A37}$$

With respect to modular transformations, two useful identities for these Θ-functions are

$$\Theta\begin{bmatrix} r/k \\ 0 \end{bmatrix}\left(x\middle|k(\tau+1)\right) = \mathrm{e}^{\mathrm{i}\pi r^2/k}\, \Theta\begin{bmatrix} r/k \\ 0 \end{bmatrix}\left(x\middle|k\tau\right) \qquad \text{only if } k \text{ is even} \tag{A38}$$

$$\Theta\begin{bmatrix} r/k \\ 0 \end{bmatrix}\left(x\middle| -\frac{k}{\tau}\right) = \frac{1}{k}\,(-\mathrm{i}k\tau)^{1/2}\, \mathrm{e}^{\mathrm{i}\pi\tau r^2/k} \sum_{r'=0}^{k-1} \mathrm{e}^{2\mathrm{i}\pi r r'/k}\, \Theta\begin{bmatrix} r'/k \\ 0 \end{bmatrix}\left(-\tau x\middle|k\tau\right). \tag{A39}$$

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